FACULTY OF SCIENCE EDUCATION

EXPLORING STUDENTS’ UNDERSTANDING OF THE CHAIN RULE IN TEACHERS’ COLLEGES IN ZIMBABWE: A CASE OF FIRST YEAR STUDENT TEACHERS AT MASVINGO TEACHERS COLLEGE

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This project was submitted in partial fulfilment of the requirements of the Master of Science Education Degree in Mathematics

Faculty of Science Education

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APPROPRIAL FORM

The undersigned certify that they have read and recommended to BINDURA UNIVERSITY OF SCIENCE EDUCATION for acceptance:

The dissertation entitled “EXPLORING STUDENTS’ UNDERSTANDING OF THE CHAIN RULE IN TEACHERS’ COLLEGES IN ZIMBABWE: A CASE OF FIRST YEAR STUDENT TEACHERS AT MASVINGO TEACHERS COLLEGE” submitted by Nyandoro Tendai in partial fulfilment of the requirements for a degree of Master of Science Education in Mathematics.

Supervisor(s): ............................................................................................................

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Date: ............................................................................................................................

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DECLARATION

I declare that: Exploring Students’ understanding of the chain rule in Teachers’ Colleges in Zimbabwe: A case of first year students at Masvingo Teachers is my own work, has not been submitted before for any degree or examination in any other University and that all the sources used or quoted have been indicated and acknowledged as complete references.

Name: Nyandoro Tendai

Signed……………………..

Date: June 2015
DEDICATION

I dedicate this work to my wife Shylet and children Tino, Tana and Taku.
ACKNOWLEDGEMENTS

I would like to express my utmost and sincere gratitude to all those who helped and advised me in pursuing this work. I am greatly indebted to Mr Chagwiza my supervisor for his untiring inspiration, mentoring and guidance throughout my research. He gave me tremendous encouragement whenever the task seemed beyond capability. I also wish to acknowledge the considerable help I got from colleagues and friends who also willingly and generously participated towards the success of this project. Of particular mention are Chidoko Clainos and Maushe Doubt. Finally my sincere gratitude is extended to my family for always being there for me, bearing with my busy schedule during the course this study.
ABSTRACT

The main purpose of this study was to explore student teachers’ understanding of the chain rule concept and was based on APOS theory. APOS is an acronym for action, process, object and schema and is constructivist theory that describes the cognitive structures that are used by learners to construct knowledge through action, process, object and schema. The study used qualitative methodology and data was collected using the instruments: questionnaire, written tasks and interviews. They were administered on a group of 42 first year student teachers majoring in Mathematics at Masvingo Teachers College. The instruments were designed to elicit the students’ understanding of the chain rule. Items on the written tasks, questionnaire and interview guide were on exploring the understanding on composition of functions, differentiating and application of the chain rule. Collected data was presented in descriptive form and selected tasks were used to see how the students formed their mental constructs to build mathematical knowledge. There was evidence that students experience difficulties in understanding the chain rule concept. Most students were seen to be operating at the action and process stages of APOS levels. Students deliberately avoided Leibniz technique and preferred the straight form techniques to differentiate functions. Understanding of the chain rule concept depends on understanding the composition of functions. For real understanding of the chain rule students should have appropriate mental structures. There is need to help students make mental constructions that assist in understanding the chain rule. The instructional strategies used in class should also incorporate ACE (Activities, Class discussion and Exercises) teaching cycle. This is a pedagogical strategy that promotes collaborative learning in a constructive way and facilitates teaching and learning of the chain rule. Students should be taught so that they can go beyond the process level to avoid a repeat of the misconceptions held by those in the historical learning.
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CHAPTER ONE

PROBLEM AND ITS CONTEXT

1.1 Introduction
Calculus is a field that presents a lot of problems and has been presenting challenges to many students in many tertiary institutions. Students often have problems in understanding the chain rule concept. It is imperative to undertake a study to find out how learners conceptualise the chain rule and how they can learn and apply this concept effectively.

This chapter is concerned with a presentation on how the research process unfolded. The background and purpose of study is presented in detail. Next in discussion is the statement of the problem, significance of the study, research questions as well as the nature of mathematics. Summary of the successive chapters is also presented.

1.2 Background of study
In calculus, the chain rule, which is one of the techniques of differentiating functions, is one of the most complicated concept in differential calculus. In most teachers’ colleges the topic Differentiation is one of the topics that is introduced to students in their Main study syllabus. A large number of students who join and choose to do mathematics receive inadequate mathematics education and join the college mostly underprepared for the study of differential calculus. Learners have a poor background and join college underprepared for the study of the topic. In addition to this, differentiation- the chain rule in particular, is not part of the Ordinary level Zimbabwe Schools Examination Council syllabus. Most of our students join college on the basis of five Ordinary level subjects that include a pass in mathematics at grade c or better. Most of these students would not have been exposed to differential calculus in their previous studies hence the poor mathematics background in differential calculus that will compromise their conceptual understanding of the chain rule.

A brief survey at Masvingo Teachers’ College, showed that many students have difficulties in understanding the concept of the chain rule. This could be attributed to poor Calculus background by a majority of them. As already pointed out the majority of the students would have taken the Zimbabwe Schools Examination Council Ordinary level syllabus and it is these students who struggle with conceptualising and applying the chain rule concept. This trend was also observed by Orton, (1983) cited in Jojo, (2011) who also showed that students:
• Had problems in understanding the meaning of the derivative when it appeared as a fraction or the sum of two parts;
• Had problem in applying the chain rule for differentiation;
• Had little intuitive understanding of the derivative and
• Had misconceptions about the derivative.

Many lecturers also admitted that they are less comfortable with teaching the topic on differentiation since their degrees are aligned to Statistics and some acknowledged that they were less conversant with its application. Although mathematics is chosen and done as a major subject of specialisation, the content covered is not in detail. Most of the work covered is Advanced level content since majority of the students are ordinary level graduates who do not have a healthy and strong differential calculus background. The college mathematics syllabus though, dictates that content to be covered must be up to first year degree level, this is not done. Topics such as differentiation, integration, functions and series are only just taught as broadening component or enrichment purposes. Little or no focus is given to preparing for further studies let alone conceptual understanding. In most cases students are only taught the basics of those topics and mostly just up to Advanced level content.

The college mathematics syllabus seeks to help students to just understand the concept of a derivative and its application to functions. It also requires students to define the derivative as a limit of the quotient and understand the relationship between differentiability and continuity. They are also expected to find the derivative at a point and find the tangent as well as the normal lines to the curve at a point. In addition students are expected to demonstrate proficiency in application of derivatives, that is, find the derivatives of sums, products, and quotients of functions. Syllabus also seeks to help students to be able to solve separable differential equations, that is, the application part.

Also time allocated to the teaching of the topics is relatively little, just about 4 to 5 lectures per topic. This is relatively a short time for conceptual understanding of the basic content. This little time spent on teaching the topics is a contributory factor in the weak preparation which consequently leads to lack of appreciation and understanding of differentiation concepts such as the chain rule in calculus.

There also has been a paradigm shift from the usual 3-3-3 programme to the new 2-5-2 programme. In the former students would spent three terms in college doing their first year,
three terms out on teaching practice and finally three terms back in college. The new programme entails only two terms in college and five terms on teaching practice before coming back to college for the final two terms. So what this literally mean is that these students have little time to interact with lecturers in college and this implies students have little time to learn the concepts. The topic is as a result not covered in detail.

So in an attempt to prepare college students with strong mathematical base that will enable them to take maths confidently for further studies, differentiation should be one the many topics that is taught to them in such a manner that they should see and make sense of it as alluded to by Parameswaran, (2007). Students must not just have procedural understanding through rote learning of procedures and symbol manipulation which is not supported by conceptual understanding. There are many researches which confirms the chain rule as a concept that causes difficulties for many students (Wangberg et al, 2010 cited in Siyepu, 2013) and so the college students at Masvingo Teachers’ College were no exception.

1.2.1 Students’ understanding of the chain rule

The students’ difficulties discussed above offer some insights into understanding misconceptions and errors which are committed by students when working some problems in differential calculus. Mundy (1984) observes that calculus students have a tendency of operating at a rote level of procedures and symbol manipulation that is not supported by an understanding of the concept involved. Consequently they fail to use calculus strategies when dealing with non-routine problems, (Clark et al, 1997). Barnes (1998) cited in Siyepu (2013) suggest that students should not be taught rules for differentiation until they have developed some understanding of what a derivative is, as well as a familiarity with the relationship between a function and its derivative. Here she is advocating for a sound base of differentiation and functions as she further elaborates that students should explore techniques on how to find and investigates derivatives of different functions.

1.2.2 Perceptions of understanding mathematical concepts

Understanding refers to how knowledge is organised in someone’s memory or head. Borowski & Borwein (1989) view understanding as the ability to perceive the meaning of something. They further explain that conceptual understanding is the ability to mentally know how a concept operates. It is evident in what learners do when given a mathematical task to do. When given a task they can be judged whether or not they have understood basing on their performance. Learners can display conceptual understanding which can be referred to as
an integrated and functional grasp of mathematical ideas. When learners have conceptual understanding they know more than isolated facts and methods. They also understand why a mathematical idea is important and even the kinds of contexts in which it is useful. In addition to that, they have their knowledge organised into a coherent whole which will make it possible to learn new ideas by connecting those ideas to what they already know. Conceptual understanding also supports retention and facts. Methods that are learnt with understanding are so connected so that they are easier to remember and can also be easily reconstructed when forgotten. If students are exposed to a method and get to understand it, they are likely to remember it correctly when called upon to. When students have conceptual understanding they can even try to explain a method to themselves and try to correct it if necessary (Jojo, 2011).

Duffin & Simpson (2000) cited in Jojo, (2011) identified three components of understanding as:

- The building-which is the formation of connections between internal mental structures at any particular time;
- The having- which is the state of these connections at any particular time;
- The enacting understanding is the use of the connections available at a particular moment to solve a problem or construct a response to a question. Jojo, (2011) observes that this is the type of understanding that maybe visible from in students’ work when responding to any given mathematical task.

According to Duffin and Simpson cited in Jojo (2011), the breadth of understanding is said to be determined by a number of different possible starting points that the learner may be having as he solves a particular problem. For instance a learner may first met functions in the form \( y = 2x - 1 \) where \( y \) value is obtained by doubling \( x \) and then subtracting one. Such an experience help the learner to develop conceptual understanding which identifies functions as formulas in which values of \( x \) are inputed to calculate the value of \( y \). The depth of understanding may be seen in the way the learner unpacks the stages of a solution by referring to other related concepts. So it is the way the learner unpacks the breath stage of his solution in more detail by referring to more concepts that demonstrates the depth of understanding. This shows that the learner would be having a deeper understanding of the problem to be solved unlike in cases where the structures will be manipulated just by
applying a rule with no explanations at all. Jojo, (2011) suggest that the understanding of a mathematical concept can be explained by adopting the APOS theory.

In this research an attempt is being made to explain the understanding of the chain rule concept with the help of APOS theory. According to the theory, to understand a mathematical concept one must begin with manipulating previously constructed mental or physical objects in his or her mind to form actions. These actions would then be interiorised to form processes which are then encapsulated to form objects. The objects can then be de-encapsulated back to the processes from which they are formed and finally would be organised into schemas, (Jojo, 2011). So the understanding of the chain rule here is explored through the development of schema relevant to it.

1.2.3 Learning in mathematics
There are different frameworks on how students learn mathematics that have been suggested. Piaget cited in Barker and Czarmocha (2002) talks about conceptual and procedural learning of mathematics. The learning process and the process of understanding is an ongoing process that leads to full understanding according to Engerbretcht, Harding and Potgiettter, (2010). They further suggest that the process of understanding new mathematics occurs in layers in which with very layer the student understands deeper. They further claim that for learners to gain deeper understanding, they have to repeatedly expose themselves to the intended concept. In addition they suggest that mathematics learning consist of two processes which are first time exposure and consolidation process. Real learning and understanding comes about by doing more problems of a certain type and this also brings repeated exposure and deeper understanding. Repeated exposures are necessary for students operating in the action stage of the APOS theory, with regards to understanding the chain rule, Jojo (2011).

Mathematical knowledge is acquired through applying mental structures to make sense of a concept to be learnt. The nature of the mathematical knowledge and how it is acquired is grounded in the APOS theory. Learning of mathematics concepts is done indirectly. Dubinsky (2010) cited in Jojo (2011) is of the opinion that appropriate mental structures for a given mathematical concept leads to easy learning of the concepts while their absence would make learning impossible. The teaching strategy to be employed should aim at assisting learners to build relevant mental constructions and the APOS theory presents the constructions as actions, processes, objects and schemas.
1.2.4 What is a Genetic Decomposition?
A genetic decomposition can be defined as a hypothetical model that describes the mental structures and mechanisms that a student might need to construct in order to learn a specific mathematical concept, according to Arnon et al, (2014). It consists of a description of the Actions that a student needs to perform on existing mental Objects and continues to include explanations of how these Actions are interiorized into Processes. While at this point, the concept is still seen as something one does. The Process is encapsulated into a mental Object so as to be conceived as an entity and something that can be transformed. It is entirely possible that a concept may consist of several different Actions, Processes, and Objects. A genetic decomposition may include a description of how these structures are related and organized into a larger mental structure called a Schema. The genetic decomposition also explains whatever is known about students’ expected performances that indicate differences in the development of students’ constructions.

It may also be added that more to describing how a concept might be constructed mentally, a genetic decomposition might include a description of prerequisite structures an individual needs to have constructed previously, and it might explain differences in students’ development that may account for variations in mathematical performance. This implies that a genetic decomposition is a model of the epistemology and cognition of a mathematical concept, according to Roa-Fuentes and Oktac (2010) cited in Arnon et al (2014).

Arnon et al (2014) point out that a genetic decomposition can act as a lens, analogous to a diffraction grating that researchers use to explain how students develop, or fail to develop, their understanding of mathematical concepts. For instance, when given a task, an individual may perform the task correctly, another may have problems, and still another may completely fail. As observed by Springer (2000) also cited in Arnon et al (2014), when structures involved in learning a particular concept are detailed, a genetic decomposition can help an instructor to uncover sources of difficulty that arise in the learning process. The genetic decomposition may be used to explain discrepancies in performance. The one who succeeds may give evidence of having successfully made one or more of the mental construction(s) called for by the genetic decomposition. The student who shows limited progress may show evidence of having begun to make the construction(s). The student who fails may not have made the construction(s) at all or may give evidence of having been unsuccessful in having made the necessary construction(s).
If the differences in student performance cannot be explained by the genetic decomposition, then that would be implying that the genetic decomposition needs revision. So in this particular case, the genetic decomposition guides the analysis as well as maybe pointing out on the gaps in the researchers’ understanding of how the concept develops in the mind of the individual. Whichever way, Arnon et al (2014) indicate that a genetic decomposition is a tool by which researchers attempt to make sense of how students learn a particular mathematical concept and to explain the reasons behind student difficulties. Genetic decomposition can also help to guide the design of an instruction by providing a description of how a concept might develop in the world of an individual. For the chain rule genetic decomposition may start with students understanding the composition of functions. These would then be transformed into one composite functions. So learners also need to have understood how to differentiate composite functions. A coordination of these two processes leads one to get the derivative of the composite functions. This would then be encapsulated to using the chain rule in differentiation.

1.3 Statement of the problem

Students are experiencing difficulties in learning derivatives and understanding the concept of the chain rule in differentiation. This could be due to the fact that they are introduced to the chain rule as merely a rule that should be applied with no justification for procedure. It could also be due to the fact that they lack certain mental constructions when they work on the chain rule problems. When exposed to the technique, they just cram how the method is employed and used as a means of getting an answer. The students just seem to be in preference of procedural methods rather than conceptual understanding as they deal with problems in calculus. The lecture-recitation method used by the lecturers also do not help matters either. So lack of mental structures hamper students’ understanding of the chain rule.

This research seeks to explore students’ understanding of the chain rule as they learn derivatives in tertiary institutions particularly at a Teachers’ College in Masvingo.

1.4 Objective of the study

- To find out how students learn the chain rule;
- To find out which APOS levels are displayed by students to understand the chain rule;
- To come up with a genetic decomposition of the chain rule.
1.5 Research question
How do students understand the chain rule?

1.5.1 Sub research questions
- What mental structures does the student need in order to understand the chain rule concept?
- What APOS levels are displayed by students when solving problems involving the chain rule?
- How do students’ mental constructions of the chain rule compare with historical understanding of the chain rule?

1.6 Significance of the study
This study shall be significant to mathematics educators as they need to understand how college students construct their mathematical knowledge when learning the concept of the chain rule so as to promote its conceptual understanding. Mathematics educators also need to understand the mental constructions students go through as they are learning the chain rule in differential calculus so as to apply them in teaching. It will also help them plan and scheme effecting the constructions. The study also contributes to a constructivist approach to learning mathematics which helps in improving the understanding of some mathematical concepts and improving on the mathematics results at all levels of mathematics education. It also provides a base for similar studies in other primary teacher training colleges. Fellow researchers also benefit by using it as reference material or by way of raising areas of further research emanating from it. More so it adds on to the researcher’s professional and intellectual maturity as an academic. It is also an addition to academic literature.

1.7 Delimitations of the study
The research was done in Masvingo urban at Masvingo Teachers’ College which is a primary Teachers’ training College situated about 7 km from the city centre. It focused on student teachers at the college where the researcher works as a lecturer in the Mathematics Department. The participants were first year students specialising in mathematics as main study. Almost every group of maths students has shown that they have difficulties in understanding the chain rule concept. So there was need to make an exploration of the concept since it was a perennial problem. The concept under study was chosen because it is covered at this level as prescribed by the syllabus. The concept is also relevant to main study and not to applied maths. This was done for convenience and for financial reasons.
Consequently, the results of the study were peculiar to the college since it was a case study which may not be generalizable.

1.8 Limitations of the study

The following aspects may limit the researcher’s efforts in executing the research:

Cooperation from students as they would be too busy with other lectures that will limit their availability and collection of data. The effect of this may be curbed by obtaining the necessary permission from the college authorities to carry out research in the college. The researcher may also be affected by employment obligation as he is a fulltime lecturer and so that would affect time and effort to carry out the research. This effect may be curbed by applying for leave for the period of data collection, which the researcher did. The researcher would also ask a colleague to do the teaching on his behalf while he sits and observes the students learning. Thus more time to observe how students are interacting and going about mental constructions of the chain rule and the genetic decomposition. The researcher and the colleague will agree on what and how to teach and with what method to use.

1.9 Assumption of the study

The following assumptions were made in this project:

- Data collected accurately is representing the area under study

1.10 Definition of terms

The key terms in the study are defined under this section and they refer to the meaning given whenever they are used in the study.

*Exploration-* is the activity of searching and finding out about something, (The Dictionary of English language). In this context it is the investigation or study of the chain rule;

*Student-* somebody who is studying at college, (Oxford Dictionary);

*Understanding-* refers to the ability to know the ways or workings of ….being able to justify procedures used or state why a process works, (Borowski & Borwein. 1989);

*Conceptual understanding-* refers to learning that involves understanding and interpreting mathematical concepts and the relationships between the concepts. In this study it refers to
the ability to mentally know how the concept of the chain rule operates (Borowski & Brwein. 1989);

*Calculus*—refers to a branch of mathematics developed by Newton and Leibneiz that concerns the rate of change of the dependent variable and the slope of a curve, (Augad, 1987). In this study it is the study of the behaviour of functions in mathematics and involves some operations in differential and integral calculus;

*Chain rule*—is a formula or technique for computing the derivative of the composition of two or more functions. If $f$ is a function and $g$ is a function, then the chain rule expresses the derivative of the composite function $f \circ g$ in terms of the derivatives of $f$ and $g$;

*APOS*—refers to the main mental mechanisms for building mental structures of actions, processes, actions and schemas.

### 1.12 Organisation of study

The organisation of the study is as follows: Chapter 1 is on the introduction and background of the study. Chapter 2 is on literature review as well as the theoretical framework of the study with Chapter 3 focusing on the research methodology. Chapter 4 focuses on data presentation, analysis, interpretation and discussion. Finally Chapter 5 is on summary, conclusion and recommendations.

### 1.13 Summary

In this first chapter the researcher has given an overview on the background of understanding of the chain rule and at a particular teachers’ college in Zimbabwe and the problems that students have as they learn the concept of the chain rule, as seen by different authorities. Focus was on the background of the study, the statement of the problem, the research question and guiding sub research questions. The delimitations and limitations of the study, operational definitions and ethical considerations were also looked into.
CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction
This chapter will start by a review of theory that informs this study, then some relevant studies on teaching and learning of the chain rule are cited and discussed. It will also investigate some possible answers to the research questions that have been put forward.

2.2 Theoretical framework
This research is based on APOS theory (Dubinsky & Mcdonald, 2001). The theory proposes that for an individual to make sense of a given mathematical concept, he or she must have appropriate mental structures. These mental structures refer to the likely actions, processes, objects and schemas that are required to learn and understand the concept. As noted by Maharaji (2013), research based on this theory requires that for a given mathematical concept, the likely mental structures have and need to be detected. After this detection, then the suitable learning activities should be designed to support the construction of those mental structures.

Asiala et al (1996) cited in Maharaj, (2013) proposed a framework for APOS theory based on research and curriculum development in undergraduate mathematics education. The framework consisted of theoretical analysis, Instructional treatment and observations and assessment of student learning. They suggest that APOS theory based research should be done according to the following paradigm:
In this paradigm, theoretical analysis occurs relative to the researcher’s knowledge of the concept under study and knowledge of the APOS theory. The analysis assists to predict the mental structures that are needed to learn the intended concept. So for a given mathematical concept, the theoretical analysis then informs the design and implementation of instruction. The theoretical analysis also informs and guides the collection and analysis of data, (Maharaj, 2013).

2.3 Description of the APOS Theory
APOS is an acronym that stands for action, processes, object and schema. This is a theory that describes how mathematical concepts can be learned and its roots are grounded in the work of Jean Piaget. Dubinsky (1984) first introduced the fundamental ideas of the theory. The theory looks at models of what goes through the mind of an individual when he or she is trying to learn a mathematical concept. It is a framework that is used to explain how learners mentally construct their understanding of mathematical concepts. According to Amon et al (2014) APOS- based research and curriculum development focus mainly on learning mathematics by students and it is a constructivist theory. APOS uses genetic decomposition as one of the tools used in APOS –based research and curriculum development. According to
the cognitive perspective, a mathematical concept is framed in terms of its genetic
decomposition. Genetic decomposition is hypothetical model of mental constructions that a
student may need to make in order to learn a particular mathematical concept. It is a
description of how a particular mathematical concept, such as the chain rule, is constructed in
an individual’s mind. Learners make sense of mathematical concepts by building and using
certain mental constructions. In APOS theory these are called the stages in the learning of
mathematical concepts (Piaget & Garcia, 1983).

According to Maharaji (2010), APOS Theory and its application to teaching practice are
based on two general hypotheses developed to understand the ideas of Jean Piaget. In some
studies such as one by Weller et al., (2003), it is reported that these ideas were recast and
applied to various topics in post-secondary mathematics. Piaget investigated the thinking of
adolescents and adults, including research mathematicians. Those investigations led him to
reveal some common characteristics, specifically certain mental structures and mechanisms
that could guide concept acquisition (Piaget, 1970). APOS theory and its application to
teaching practice are based on the following assumptions which were seen to be guiding its
application to the teaching practice (Dubinsky, 2010):

- **Assumption on mathematical knowledge**: An individual’s mathematical knowledge
  is his/her tendency to respond to perceived mathematical problem situations and their
  solutions by reflecting on them in a social context, and constructing or reconstructing
  mental structures to use in dealing with the situations.

- **Hypothesis on learning**: An individual does not learn mathematical concepts
directly. He/she applies mental structures to make sense of a concept (Piaget, 1964).
  Learning is facilitated if the individual possesses mental structures appropriate for a
given mathematical concept. If appropriate mental structures are not present, then
learning the concept is almost impossible.

What these assumptions are implying is that the goal for teaching should consist of strategies
for helping students construct the right mental structures, and guiding them to apply these
structures to construct their understanding of mathematical concepts. The APOS theory
emphasises that conceptual formation works in stages and the construction of a complete
mental structure operates through, the mental structures: actions, processes, objects, and
schemas.
2.3.1 Description of the structures

*Action*- is a transformation of objects which are seen by an individual as essentially external and as requiring step by step instruction on how an operation is performed. A transformation is first seen or conceived as an action when it is a reaction to stimuli which a learner sees as an external. It needs specific instructions and the need to do each step of the transformation clearly. For instance, a learner who requires an explicit expression to think about the derivative function, \( f'(x) \), where, say given \( f(x) = x^5 \), if the learners can do little more than just perform the action \( f'(x) = 5x^4 \), then he or she is considered to be having an action understanding of the derivative of a function (Maharaj, 2013).

*Process*- refers to a mental construction that are made by an individual when an action is repeated and reflected upon it. When an individual repeats an action, this action may be interiorised into a mental process. Maharaj (2013) refers to a process as a mental structure that performs the same operations the action, but wholly in the mind of the individual. The learner can just imagine doing the transformation without having to execute the steps explicitly. For a student with a process understanding of the derivative function, take for example, \( g(x) = (x^3 + 1)^2 \), This individual can construct a mental process which could include that \( g(x) \) should first be simplified by squaring the term \( x^3 + 1 \) before differentiating by the rule, that the derivative of a sum of function is the sum of the individual derivatives of functions. An individual can think of performing the same kind of action, though this time no longer with the need of external stimuli. Dubinsky (2001) says that an individual might just think of performing a process without actually doing it. He can think about reversing it and comparing it with other processes.

*Object*- this is a construction from a process when the individual becomes aware of the process as a totality and realises that transformations can act on it. If the learner can realise this and can actually construct the transformations, then it can be said that the learner has encapsulated the process into a cognitive object. When finding, say, the derivative of functions an individual, may confront situations that may need him or her to apply different actions and processes. Amongst the various actions, could be that the learner thinks about a function as the composite of two functions. For example, consider the function \( h(x) = (x^3 + 1)^2 \) is the composite of the functions \( f(x) = x^{20} \) and \( g(x) = x^3 + 1 \), since we have \( h(x) = f(g(x)) \). According to Maharaj (2013), the derivative \( h(x) \) is found by first conceptualising it as an object which is made up of the composite of two functions. She then
suggests, the process understanding for finding derivatives should be encapsulated in the chain rule context in order to find the derivative $h'(a)$.

Schema- this now is an individual’s collection of actions, processes and objects together with other schemas that may be linked by some general principles to form a framework in the individual’s mind. These need to be organised into and connected into a coherent framework called a schema. The coherence is due to the fact that it provides an individual with a way of deciding, when presented with a mathematical problem, whether the schema applies or not. Dubinsky (2001) maintains that this is the framework in an individual’s mind that may be brought to bear upon a problem situation that involves the concept.

In this research construction of knowledge was analysed through Reflective abstraction using APOS theory as suggested by Dubinsky (2001).

The reflective abstraction has two components which are:

- a projection of existing knowledge onto a higher plane of thought and
- the reorganization of existing knowledge structures, according to Dubinsky (1991).

Reflective abstraction is therefore a process of construction. Dubinsky identifies five kinds of construction in reflective abstraction:

1. Interiorisation: Actions conceived structurally as objects are interiorised into a system of operations. According to Sfard (1991) during the phase of interiorisation a student becomes familiar with a process and can carry it out through mental representations. Different mathematical notions can be seen either structurally as objects and procedurally as processes and Sfard describes the route from processes to objects as involving interiorisation, condensation and reification

2. Co-ordination: At this level of abstraction two or more processes are co-ordinated in order to form a new process, for instance, the chain rule technique for differentiation relies on the co-ordination of composition of functions with derivatives as propounded by Bowie (2003). This was referred to by Sfard (1992) as condensation where there is a gradual quantitative change in which a sequence of mathematical operations is
dealt with in terms of ‘input and output without necessarily considering its component step’.

3. **Encapsulation**: This is a stage where the construction of mathematical understanding proceeds from one level to the other. New forms of the process are built drawing from the previous ones to form an object. This is in support of Sfard’s notion of reification which involves the student’s “mind’s eyes ability to envisage the results of the processes as permanent entities in their own right”, (Sfard & Linchevski, 1994 cited in Jojo, 2011). This process takes place when a student is able to separate the understanding from the processes that produced it and see it as an object. As suggested by Dubinsky et al, (1989) the composition of functions can be seen as a binary operation that acts on two functions, considered as objects to form a third one. They then suggest that the student has to unpack these functions, reflect on corresponding processes, and interiorise them. The two processes can then be encapsulated into an object which now is the function that results from the composition. According to Dubinsky this process is the most complicated one than simple substitution and concurs that students have a problem with the chain rule for differentiation where it is necessary to co-ordinate the view of composition of the function with the notion of derivative (Jojo, 2011).

4. **Generalisation**: When we have an existing schema that is applied to different contexts then we have generalisation taking place. This can happen for instance, when the student is able to see that after finding the derivatives of the various functions in a composition, they realise that now they have to be multiplied to put the chain rule into application.

5. **Reversal**: This is a stage where new process can be constructed by means of reversing the existing one in the process say of using the chain rule. For an interiorised process we can think of it as a reverse. For instance the reversal of the chain rule could be used in working an integration problem which can be thought of as a reverse process of differentiation.
Maharaji (2013) notes that explanations that are offered by an APOS analysis are limited to descriptions of the thinking which an individual might be capable of. She further observes that, it does not necessarily mean that if an individual possesses a certain mental structure, then he or she will apply it in a given situation as this depends on other factors. The main objective of an APOS analysis is to point to possible pedagogical strategies.

2.4 The ACE Teaching Cycle
This is also an approach that is based on APOS theory. The hypothesis of teaching and learning is a repeated cycle that consists of three components which are Activities (A), Class discussions (C), and Exercises (E) which are done outside of class, Asiala et al (1996) cited in Maharaji (2013).

The first step of the cycle is activities which are designed to foster the students’ development of the mental structures that are called for by an APOS analysis. The teacher while in the classroom guides the student to reflect on the activities and all its relation to the mathematical concepts that are being studied. Learners do this by performing given mathematical tasks. After performing the tasks they discuss their results and listen to explanations from fellow students or the teacher for the meanings of what they will be working on. Homework exercises are prepared and given to help consolidate understanding of problems. They also reinforce the knowledge obtained in the activities and classroom discussions. The students then will apply this knowledge to solve similar problems of the topic being studied. This approach is effective in helping students make mental constructions and learn mathematics as supported by several other researches (Weller et al, 2003).

2.5 Functions
Functions form an important component in differential calculus. Leinhardt et al (1990) have presented some research papers focusing on functions and graphs. Other researchers such as Tall (1992) and Even (1993) cited in Contrill (1999) note that a function is a dichotomous concept which is seen by students as a process or an object. Harel & Dubinsky (1992) also contends that when students learn functions they start with an action conception of function before moving to process and object. What is common from these reviews is that the concept of functions is important in mathematics especially with regards to understanding the chain rule but is not well understood by most students. Contrill (1999) observes that there is not much study done yet regarding the students’ understanding of composition of functions and particularly no much study on understanding the chain rule.
2.6 The chain rule
The chain rule is a technique that is used to find the derivatives of composite functions. Kaplan (1984) points out that the chain rule is a function of functions. A Composite function is referred to as a function that is composed of two or more functions or a function whose variable is another function. For instance if \( f(x) = \cos(x) \) and \( g(x) = x^2 + 1 \), then \( f(g(x)) = \cos(x^2 + 1) \). The second function is an example of a composite function. We have the lone variable \( x \) from the first function replaced by \( x^2 + 1 \), which is also another function on its own. Such a function is hard to differentiate on its own without the aid of the chain rule. According to Jojo, (2011) the chain rule is an important concept that underlies in many applications of calculus that include implicit differentiation and solving related rate of change problems. In addition it is applied in the fundamental theorem of calculus and solving differential equations.

2.6.1 The Formula
When we have two functions \( f \) and \( g \), the differentiation of the Composition of \( f \) and \( g \), is given as:

\[
\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).
\]

What this this rule is saying is that if we have a composite function and need to take its derivative, all we have to do is to take the derivative of the function alone, then we multiply it with the derivative of the smaller function. However there is a condition that needs to be satisfied before one can use the rule, that is, both the functions must be differentiable at \( x \). So this means that if one of the functions is not differentiable then the chain rule cannot be used.

The chain rule helps us to find the derivative of functions that are compounded inside of one of another. The chain rule is important in differential calculus and has been studied in mathematics educational research despite its difficulty for students (Gordon, 2005; Uygur & Ozdas, 2007). Students had difficulties that include the inability to apply the Chain rule to functions and also with composing and decomposing functions as observed by Clark et al, (1997) and Cottrill, (1999).

In my Experience at Masvingo Teachers’ College students experience most problems in the topic differentiation where the chain rule is a concept that is used in problem solving. Compared to other topics such as circle geometry, trigonometry, vectors and complex
numbers, students’ performance showed difficulties in understanding the chain rule. It was as a direct result of this that the researcher sought to find out how learners conceptualised the chain rule and try to see how they can effectively learn the concept. So the research focuses on helping the students learn and understand the chain rule at Masvingo Teachers’ College. It will also inform the researchers teaching of the topic as well as effective learning of the concept by the student.

### 2.6.2 Teaching the chain rule

When we ask students to perform a procedure such as differentiating a composite function, students can often an example and get a correct answer without showing an understanding of how and why the process works. Traditionally it has been observed that the majority of assessment in mathematics learning has been based on students’ abilities of a procedural format. This is also the case when teaching the chain rule where efforts of trying to find out the mental structures needed to understand the chain rule are not pursued. Students answer questions according to the way they would have been demonstrated without understanding how the process works, thus students are learning only by memorising operations with no understanding of underlying meanings (Arslan, 2010). While it is normal and good to give tests, exercises and even examinations, this only aims at students producing correct answers and highlights just their abilities to show how the mathematics processes work, but do not shed on the deeper understanding why. Students as asserted by Hirbert and Leferyre (1986) need to know how and why something happens in a particular way. So students need to have conceptual knowledge and this cannot be learned by rote but rather by thoughtful and reflective learning.

### 2.7. Related Studies

The chain rule is one of the most challenging concepts conveyed to students in differential calculus as noted by Gordon, (2005). Most students do not really understand its source and as a result it becomes difficult to motivate them to understand it. It is a difficult concept to express in symbols even after it is developed, and it is difficult to put into words, so much that many students cannot remember it, and so cannot apply it correctly. Swanson (2006) notes that the complexity of the chain rule deserves exploration because students struggle to understand it. Furthermore it also needs to be investigated because of its importance in the calculus curriculum. Despite its importance in calculus and its difficulty for students, the chain rule has been scarcely studied in mathematics educational research (Clark et al, 1997; Gordon, 2005; Uygur & Ozdas, 2007; Webster, 1978). Student difficulties include their
inability to apply it to functions and also with composing and decomposing functions as observed by Clark et al, (1997) and Cottrill, (1999). Some previous researches done emphasized the importance of composition of function in the understanding of the chain rule, but these have not been able to say more than that, (Clark et al, 1997; Cottrill, 1999; Hassani, 1998). In this research focus will be on how students understand the function composition as seen through the chain rule problems by using functions that are familiar, and unfamiliar to them. Uygur & Ozdas (2005) asserts that the derivative is an inherently difficult concept for many students. Jojo,(2011) agrees that this becomes clear when the function considered is a composite function and students’ difficulties increase and get worse (Tall 1993). Students also have problems in using the Leibniz notation, \( \frac{dx}{dx} \) whether it is a fraction or a single indivisible symbol. According to Jojo (2011) this causes serious conceptual problems, but this notation is indispensable in calculus. According to Tall (1993), cited in Jojo (2011) the difficulty with the notion of the chain rule is the dilemma of whether \( du \) can be cancelled in the equation: \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \). It is clear from the above discussion that, many functions have simple expressions for their derivatives while composite functions need the use of the chain rule for differentiation. Functions with complicated expressions have clear formulae for derivatives. It was due to the development of formulas and techniques such as the chain rule that facilitated the calculation of the derivatives of functions and motivated the use of the name calculus for this mathematical discipline. These difficulties experienced by students in using the chain rule probed my research in this direction as outcomes from this study will also inform better instructional techniques.

2.8 Recent studies using APOS
Brijlall and Maharaji (2010) investigated fourth-year undergraduate teacher trainee students’ understanding of monotonicity and boundedness of infinite real sequences, using APOS theory. They prepared worksheets which were based on examples and non-examples approach as a way of promoting collaborative learning. In this instance they wanted to find out the use of a structured worksheet design applying the APOS theory, to promote collaborative learning, and how it could influence the construction of concepts in real analysis (Jojo, 2011).

They found out that prepared structured worksheets encouraged group work and also promoted a conducive working environment to reflect abstraction. In addition students demonstrated an ability to apply symbols, language and mental images to construct internal
processes to make sense of the concepts of monotonicity and boundedness of sequences. Students were also able to apply actions on sequences and the conceptualisation of the concept of boundedness of sequences and monotonicity facilitated the formulation of new schema which would be applied in different contexts.

In another study Brijlall and Maharaj once again explored APOS theory to develop insight into pre-Service mathematics students’ mathematical reasoning for the derivative concept. The theory was used as a guide in finding out whether pre-service mathematics students can use different interpretations of the concept of continuity correctly and effectively in their reasoning. According to Brijlall and Maharaji cited in Jojo (2011), the APOS theory provided them with an opportunity to develop the genetic decomposition of the continuity concept. In their study they concluded that participants displayed a good potential for learning the continuity concept when allowed to collaborate and work in small groups. It was also their finding that having a structured encouraged group work which fostered an environment suitable for reflective abstractions.

In a more recent study on an APOS analysis of natural science students’ understanding of the derivatives at Westville Campus of the University of KwaZulu Natal in South Africa, Maharaji (2013) observed the building block for the concept of the derivative is the function concept. As had been noted by Tall (1997), functions form a central part of the pre-calculus and calculus curriculum.

In a study of the first year calculus students prescribed textbooks at the University of KwaZulu Natal, it was revealed that, to introduce the derivative concept a good grounding in algebraic manipulations and functions are a prerequisite (Maharaji, 2013). It was also pointed out that before the introduction of the derivative concept learners should have already established algebraic manipulative skills as well as the concept of a function. It was also observed that the learners’ mental structures of functions should be at process level and even higher, since it is at this level where they predominantly rely on the use of and algebraic formulae when dealing with the function concept and ultimately the chain rule (Contrill, Dubinsky & Schwingerdorf, 1997, Akkoc & Tall, 2005).

More to that, other studies also emphasized on the importance of the function composition concept in the understanding of the chain rule- a useful technique for finding the derivative of
functions such as \( f(x) = \sqrt{x^2 - 1} \). In the problem, we can take \( f(x) \) as a composition of two functions, for example, have \( f(x) = g(h(x)) \) where \( g(x) = \sqrt{x} \) and \( h(x) = x^2 - 1 \).

According to Tall (1997), students experience difficulties when they meet up with a composite function and they are supposed to differentiate it. As a result the chain rule, becomes the hardest ideas to convey to students (Gordon, 2005, Uygur & Ozdas, 2007).

Clark et al (1997) cited in Maharaji (2013) studied students’ understanding of the chain rule and its application. They concluded that the difficulties experienced with the chain rule by a large number of students could be due to the difficulties they experience when dealing with composition and decomposition of functions. Maharaji (2013), concludes by observing that this implies the understanding of composition functions is an integral part to understanding the chain rule and is in support of other studies by Contrill (1999) and Horvath (2007).

The cited studies indicate that exploring the theory is relevant to analysing mental constructions that could be made by students in learning of various concepts such as the chain rule in calculus. In Zimbabwean Teachers’ Colleges, no study has yet been made, focusing on the exploration of the understanding of the chain rule. This was seen as a gap and this research is an attempt to address the gap.

2.9 Mental constructions needed in understanding the chain rule

For a student to get new mathematical meaning about a concept, he/she would need to construct mental representations of experiences that are of relevance to that particular concept. According to Dubinsky (1991) the student need actions, processes, objects and schemas (APOS) as mental constructions. A structured mental construction such as this, which describe the development of a concept in an individual’s mind is called genetic decomposition of the particular concept. According to the APOS theory understanding of a mathematical concept such as the chain rule, begins with manipulating previously constructed mental objects in the learners’ mind so as to form actions. These actions are then interiorised to form processes which in turn are then encapsulated to form objects. Dubinsky further says the objects are de-encapsulated back to form processes from which they were formed. Finally they are organised into schemas. Hassani (1998), cited in Jojo (2013) undertook a study to check on students’ understanding on graphical and symbolical presentations of composition of functions and the chain rule. Results of this study showed that the group of the first year graduates calculus students had little understanding of the composition of the chain rule.
Their ability to explain or apply the chain rule concept was also seen to be related to their algebraic manipulative skills. They were also seen to depend heavily on their knowledge of concepts of function as well as function composition. Swanson (2006) also observed that the chain rule concept was difficult to understand by students since they struggled to understand it and called for its exploration.

Since students experience difficulties in understanding and applying the chain rule concept, they need to be given class activities that are designed to induce them to make the suitable mental constructions as suggested by what Jojo called initial genetic decomposition (IGD). After class activities students are provided with more difficult or challenging activities which need them to organise different previously constructed objects such as functions and derivatives of composite functions into schemas which would be applied to the chain rule problems. Jojo (2013) presented what she called Initial Genetic Decomposition for describing some specific mental constructions that a student should make so as to develop a conceptual understanding of the chain rule. The following diagram illustrates this:

Figure 2.2: Initial Genetic Decomposition for mental construction of the chain rule

![Initial Genetic Decomposition for mental construction of the chain rule](source-jojo-2013.png)

Source: Jojo (2013)

In this diagram Jojo (2013) proposes that understanding begin with:

1. Students understanding the composition of two or more functions
2. Transformation of the function to one composite function;
3. Understanding of the derivative concept of the composite function and then
4. Co-ordination of the two processes to get the derivative which would be then encapsulated to using the chain rule for differentiation.
In the next illustration Jojo (2011) also tries to explain the activities that are involved in construction of the chain rule concept.

**Figure 2.3: Activities involved in construction of the chain rule**

![Diagram](image)


Here the student begins with two functions F and G which are then transformed into a single function \( F \circ G \). The transformation begins by de-encapsulation of the two functions F and G back to the process \( F(x) \) and \( G(x) \) from which it came. These two processes are then coordinated to get a process on \( F(G(x)) \) which is then encapsulated to get the object \( F \circ G \).

**2.10 How students construct understanding of the chain rule**

Burke et al (2001) alludes to the fact that there is growing research support for designing classroom instruction that looks at developing deep knowledge about mathematics procedures. Star (2000) posits that when instruction is focused only on students’ ability in skilful execution, they develop automated procedural knowledge that is not strongly connected to any Conceptual knowledge network. In the end it was noted that this instruction resulted in procedures not executed “intelligently” and systematically. Real understanding could be achieved when students were given a chance to develop a framework for understanding any appropriate relationships, by extending and applying what they knew as well as reflecting on their experiences, and allowed to make mathematical knowledge their own (Carpenter & Lehrer, 1999). In addition when mathematical knowledge is understood, that knowledge is more easily remembered and more readily applied in a variety of situations. When a unit of knowledge is part of a well-connected network of mathematical understandings, parts of the network can facilitate recall and sometimes even recreation of
other parts, and When knowledge is understood it becomes easier to incorporate new knowledge into existing networks, so that current understanding prepares for future learning (Hiebert & Carpenter, 1992).

It is of importance to make use of teaching strategies that help students to develop mathematical understanding. Brijlall & Maharaj (2009) employed the APOS theory in a study where they investigated fourth-year undergraduate teacher trainee students’ understanding of the two fundamental concepts monotonicity and boundedness of infinite real sequences. Their findings were that:

- the structured worksheets encouraged group work and fostered an environment conducive to reflective abstraction,
- the students demonstrated the ability to apply symbols, language, and mental images to construct internal processes as a way of making sense of the concepts of monotonicity and boundedness of sequences,
- the students could apply actions on objects (sequences) which were interiorized into a system of operations, and
- the conceptualization of the concept of boundedness of sequences and monotonicity enabled the formulation of new schema which could be applied in various contexts.

According to Dubinsky & McDonald (2001) cited in Jojo (2013), mathematical ideas start with human activity and then proceed to be abstract concepts. It is also vital for us to understand how the construction of concepts in the mind, can lead to abstraction of mathematical knowledge. This way of interpreting the relevant knowledge construction processes is important since it points to the contributions that are derived from analysing the APOS theory. Some of the contributions according to Jojo (2013) include:

- understanding the importance of human thought, and
- Pointing to effective pedagogy for a particular concept.

The same author, for example, says that a constructivist approach was explored in the teaching of differentiation in calculus. Under this there were classroom activities that were used that included working in groups, individual work, class discussions and some mini-lecture presentations summarising the results of the learners’ work and then providing examples on the use of the chain rule in differentiation.
2.11 Misconceptions with the chain rule
Mathematics of different times must be used together with modern Mathematics. One must appreciate and understand the differences between the mathematics of different times, for failure to do so may betray the existence of pitfalls and ill practices in the didactics of mathematics (Rodriguez & Fernandez, 2010). In informal experiments that were conducted in an introductory non-standard calculus course at the University of Puerto Rico, they report that it was noted that Students had significant difficulties in identifying the composed functions before applying the chain rule.

In differential calculus, the derivative is a basic but difficult concept for many students. According to Jojo (2011), students often encounter increased difficulty when functions to be considered for differentiation are in composite form. They also have some incorrect conceptualisation of the chain rule. In calculus this concept is considered to be one of the hardest to convey to students as alluded to by Gordon (2005). Most students do not understand where the rule comes from and as result have difficulty in symbols and construct their understanding of it. Jojo further says that it is even awkward to put it in words to the extent that many students cannot remember it, let alone apply it correctly.

Tall (1993) cited in Uygur & Ozdas (2005) indicated that the Leibniz notation causes some serious conceptual problems to students. They are faced with the dilemma of whether the \( \frac{dy}{dx} \) in the equation \( \frac{d}{dx} \frac{dy}{du} \cdot \frac{du}{dx} \) can be cancelled or not. They have a question: Is it a fraction, or a single indivisible symbol? Most students, as noted by Uygur & Ozdas (2005) find themselves within a notational and conceptual complexity. This complexity of the chain rule, calls for an exploration of its understanding since students struggle to understand it and also because of its importance in calculus.

While there is literature into understanding of the concepts underlying calculus Hassani (1998) as cited in Jojo (2011) acknowledges that there are significant gaps between students’ understanding of main calculus ideas and their ability to perform procedures based on these ideas. Jojo (2011) goes on to note that results from Hassan’s research showed that the first-year undergraduate students’ ability to apply the chain rule is related significantly to the students’ ability to manipulate algebra together with their general knowledge of function concepts and composition. Students’ mental construction of the chain rule schema is
seriously affected when students are having problems with the use of the Leibniz notation such as deciding whether \( \frac{dy}{dx} \), is a fraction or a single indivisible symbol.

2.12 Summary
In this chapter, literature on functions, composition of functions, the chain rule, how students learn mathematics, difficulties and misconceptions with the chain rule as well as recent studies on APOS were reviewed. It became clear that learners form mental constructions at different levels as defined by APOS. An interpretation of the mental construction using APOS was also done. The literature reviewed that when a student has a collection of derivative rules and understanding of compositions functions he/she would be capable of operating on mental constructions, to build on his/her mathematical knowledge. Current knowledge rests upon what was done before. The next chapter focuses on the research methodology, research design and data collection through various research instruments.
CHAPTER THREE

RESEARCH METHODOLOGY

3.1 Introduction
A review of related literature was made in the previous chapter. Also an analysis of how several authors viewed the topic was made. The present chapter is concerned with research design, sampling procedures, research instruments and data collection procedures.

3.2 Research design
Qualitative methodology was employed in this study, to collect and process data. The data were collected using written tasks, questionnaires and interviews since the study was concerned with exploration of understanding of chain rule by students. The qualitative inquiry comes in handy when the researcher skilfully devises a tool to probe deep within the minds or attitudes, feelings and reactions of the respondents.

In this research the respondents, the students were expected to express their opinions, feelings, attitudes in separate focused group interviews. The semi-structured interview approach was used to follow up key observations in detail or to allow opinions or perspectives to emerge freely (Vulliary et al, 1990).

The researcher also administered a questionnaire to students in order to collect information that the researcher wanted to support their verbal responses on their understanding of the chain rule, while maintaining anonymity. The responses from the questionnaires, written tasks and interviews were then compared to see if there was any consistency – triangulation.

3.3 Population of study
The population constitutes all the characters under investigation. According to Chiromo (2012), population refers to all the individuals, units, objects or events that would be considered in a research project. In this study the target population consisted of all the student- teachers doing mathematics as their main study and are in their first year at Masvingo Teachers’ College. This was the group the researcher was interested in gaining information and drawing conclusions. At the time of carrying out the research the group was in college and so there were no transport costs incurred on the part of the researcher. So the population comprised of the students and lectures.
3.3.1 Sampling procedure
Sampling was done purposively. Hoberg (1992) says that in purposive sampling researchers handpick the case to be included in the sample on the basis of their judgements. In this way they build up a sample that is satisfactory to their specific needs. www.Experiment.resource.com/home also posits that in purposive sampling, subjects are chosen to be part of a sample with a specific purpose in mind. Thus in this study purposive sampling was seen suitable for the first year student teachers at Masvingo Teachers’ College. Permission to conduct interviews was sought from the college authorities prior to carrying out the research.

3.3.2 Sample
A sample is a part of the population under study. It is selected from a targeted population. Best and Kahn (1993) describe a sample as a portion of the population selected for observation and analysis. Sampling is a process of selecting a number of individuals for study in a way that they represent the population. The intention of sampling is that the sample accurately represents the whole population and be representative enough so that results may not be doubted. It is usually difficult to work with the whole population, hence the choice of a sample. The main objective of drawing a sample is to make inferences about the larger population from the smaller group which should reflect the characteristics of the total population.

The Mathematics Main study students were the target population and a total of 30 first year students at Masvingo Teachers’ college were used in the sample. These were selected since they were the ones in college at the time when the other group was out on teaching practice. They were also the ones to cover the topic on differentiation which includes the chain rule concept. From the first years 24 were females while 18 were males.

The researcher also worked out with four lecturers all in the mathematics department and teaching mathematics. These were also selected since they had a fair experience of having taught both Ordinary level and Advanced level mathematics in the high schools and so knew about differential calculus. The questionnaires and interview questions were given to fellow lecturers, so that they look at them and retrieve whatever information they may require from their memory without pressure.

The researcher ensured that the sample displayed all the characteristics of the population in order to make it representative enough. Leedy (1999) asserts that , the sample should be
carefully chosen so that the researcher is able to observe all the characteristics of the total population in the same relationship that they would be observed were the researcher in fact to inspect the total population.

Table 1: Distribution of the sample

<table>
<thead>
<tr>
<th>YEAR GROUP</th>
<th>FEMALES</th>
<th>MALES</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>FIRST YEARS</td>
<td>18</td>
<td>12</td>
<td>30</td>
</tr>
</tbody>
</table>

Source: Author Compilation

3.4 Validity and Reliability

The validity and reliability of measurements made in research need to be considered.

3.4.1 Validity

According to Leedy (1993) validity is concerned with the soundness or effectiveness of a measuring instrument. The questions to ask have to do with what the instrument is trying to measure and whether in fact the instrument measures what it is supposed to measure. In this study all the questions in the tasks and questionnaires all ask about information on differentiation and the chain rule in particular. Data are valid if they provide a true picture of what is being measured. Data generated in the questionnaire and the tasks actually give relevant information on the students’ understanding of the chain rule. A valid statement should give a true measurement, description or explanation of what it claims to measure or describe. Thus, for this study, validity is referring to the effectiveness of data generation instruments like interviews, document analysis and the questionnaire in exploring understanding of the chain rule concept by student teachers.

3.4.2 Reliability

Reliability deals with the accuracy of measuring instrument. One would want to find if the same result would be obtained if the same instrument were used under similar conditions. According to Best and Kahn (1993) reliability is the degree of consistency that an instrument or procedure demonstrates. This means that whatever the instrument measures, it does so consistently. Haralambos, Holborn and Heald (2008) postulate that data are seen to be reliable if other sources using the same investigative methods will produce the same result. In this study, reliability was enhanced by creating an atmosphere that was free from distractions.
For the interview sessions and answering of the questionnaire the participants were told of the purpose of the research as well as the potential benefits of the study. A conducive environment was created for respondents to feel free during the interview sessions and completion of the questionnaire.

Validity and reliability in this research were achieved through triangulation. Three instruments were used, the questionnaire, document analysis and open ended interview. Results got using these three instruments were compared to see if the same results were obtained. The interview guides used for students consisted of open ended questions that the researcher would ask. All the respondents would be asked the same initial questions. Questions were structured in such a way that the respondent was required to give his/her opinion or feeling on a particular notion.

3.5 Research instruments
3.5.1 Document analysis
Document analysis was also engaged as a data generating tool. Neuman (2000) as cited in De vos (2003) asserts that documents are primary and secondary sources of data. So students’ work in the form of written tasks were also scrutinised to get some insight on their understanding of the chain rule concept. It was also necessary to go through the students’ written work to also search for issues related to understanding of the chain rule concept, that is, those which could not be detected in either the questionnaire or the interview.

3.5.2 Questionnaire
The researcher administered the questionnaires personally to ensure a 100% return. The researcher also had a chance to explain to the students what was not clear on the questionnaires. It was given to all 42 students. From the 42 about nine students were sampled out as three best, three average and three lowest performers in the written task and their responses were analysed. These were picked basing on their performance in the written task.

The questionnaire was used as a research instrument and was administered during the students’ regular lecture time in order to explore their understanding of the chain rule. The purpose was to gain insight into how students performed their mental constructions on the chain rule. It is important to understand how they do this so as to inform effective teaching and learning of the concept. The questionnaire is a power full tool as it enabled the researcher to gather data from the selected 42 first year mathematics group within a short space of time.
It consisted items which included functions, compositions, differentiation and application of the chain rule.

The questionnaire had some items coded A, B, C, and D where they had to choose one answer with regards to understanding of the chain rule. The questionnaire items were broken down into more direct questions and alternative answers were provided from which respondents had to choose. In some cases an item would be followed by a question where the respondent has to explain his/her choice. In cases where there was yes/no question, the respondents were asked to explain or justify their choice.

From the respondent’s responses the researcher would then make some conclusions on students’ understanding of the chain rule concepts. The researcher would also see if the results were consistent with those from other authorities.

3.5.3 Interviews
According to Schulze (2001) interviews are attempts to understand the participant’s point of view. An interview consists of the researcher asking the interviewee or respondent a series of questions and so this is a face to face interaction between interviewer and interviewee (Haralambos et al, 2008). In this study focused group interviews were conducted with students in college. The focused group interviews were chosen ahead of other data gathering devices because people are more willing usually to talk than write. One of its main advantages is that after the researcher establishes good rapport with respondent he gets the information he wants. Thus, certain confidential information may be obtained, especially that one might otherwise be unwilling to disclose in writing. Another advantage is that the researcher can explain exactly what he wants and ask follow up questions should the respondent misinterprets the original question. Interviews were effective in this research as the researcher was able to interview students in order to acquire information on the understanding of the chain rule concept. In addition interviews ensured high response rate and were also suitable for respondents who were not fluent in writing.

The researcher conducted the interview himself and picked a representative from the identified groups to interview to get more insight into how students make their mental construction in as far as understanding the chain rule was concerned. The researcher interviewed selected students for clarity and further explanations on what they had given as responses in questionnaires, written tasks and class activities. Open ended questions were
asked and these were used to extract as much information from participants as possible. They also allowed the participants to express themselves freely on their way of thinking when they responded to the questionnaire. The interviews basically were done to describe how those students constructed the concept of the chain rule

3.6 Data Collection Procedures
Data will be collected in as many forms of class activities, discussion and exercises done out of class (ACE). Other possible sources of data would be tests and homework, assignments, direct observations, questionnaires and some clinical interviews. These would be conducted on tasks that are pre-supposed to test students’ understanding of:

- Composition and decomposition of functions;
- Differentiation;
- Use and application of the chain rule in differentiating functions.

These would be administered during tutorial time held each Thursday afternoon in 1 hour slots as a teaching strategy based on APOS theory. The Thursday lecture time would be organised to reinforce learning and teaching done during the lesson on differentiation concepts in normal lectures. Students would be divided into groups, say five with the researcher supervising each of the groups. Students would be encouraged to attempt the task questions and told that it was not important whether they gave correct or wrong responses, since what would be of significance would be the procedures they would use. Interviews would also be conducted with at least one member from each group to explain the procedure they would have used.

3.7 Data Analysis and Presentation Techniques
The data obtained from the interviews was strictly from students only and no lecturers were involved. Analysis of data would be done according to the outlined research questions asked. Data would not be presented in the form of tables. Descriptive statistical analysis for this data would be used. For instance the researcher would prefer to use frequencies of occurrence, like 8 out of 10 students believed following procedural methods are better than conceptual understanding when working chain rule problems. Using percentages like 80% would give a distorted picture since the sample was very small. With this type of analysis, generalisation is limited to only to the particular group of individuals observed due to the small size of the sample.
The researcher would then compare interview responses from the selected students with what they would have said in the questionnaires.

3.8 Proposed genetic decomposition of the chain rule

To see how students were building their knowledge on the chain rule the following genetic decomposition will be used:

1. Students should show understanding of functions, that develops a process conception of a function;
2. He/she was expected to have developed a process conception of composition of functions;
3. Now in order to compute the derivatives, the student develops the conception of differentiation, which is the process;
4. After this, the student coordinates all the previously constructed schemes of function, their composition and differentiation so as to be able to define and understand the chain rule.

The student is expected to do this coordination after recognizing that the function given is a composition of two functions. It is then after that he takes separate derivatives of the functions then multiplies them to get the answer and thus using the chain rule.

3.9 Ethical considerations

It is expected that when one is engaged in research he has to pay great attention to ethical issues. Ethics as defined by Musazi & Kanhukamwe (2003) are a code of behaviour or expected behaviour of an individual or a group of people in an organisation. Ethics and morals are simply concerned with the right and wrong within the context and delimitation of one’s study.

To ensure objectivity and integrity of the study some ethical considerations had to be taken into account. Informed consent was one issue to be observed by the researcher in this study. Thus participants who were to take part in responding to the questionnaires and interviews as well as the written tasks were fully informed permission to interact with them was sought from the college authorities. Participants were told that participation was voluntary and they were free to withdraw at any time.
Participants were also assured of anonymity and confidentiality in the study. These meant issues of confidentiality are to be respected. In addition the principle of beneficence was communicated to the participants. They were informed of the potential of this research study to improve the academic arena that is, teaching and learning in colleges.

The researcher also mentioned that the information collected would only be used for academic purposes. Thus participants were assured that the exercise was a truly academic work and data collected would be used for research purposes only.

3.10 Summary
This chapter dealt with the research design, sampling procedures, research instruments, data collection procedures, validity and reliability and data analysis and presentation techniques. The design involved both quantitative and qualitative methods. The population of study was also defined and sampling procedures were outlined. The next chapter will deal with data presentation, analysis and interpretation of results.
CHAPTER FOUR

DATA PRESENTATION, ANALYSIS AND DISCUSSION

4.1 Introduction
The chapter focuses on the presentation of the generated data. Basically the data to be presented were sourced from the interview guides, written tasks and the questionnaire. The analysis and interpretation of data would be done following the sequence of research questions. Subsequently the chapter would conclude with a summary of the chapter.

In this study, qualitative methods of collecting data were used. Data gathering tools included written tasks, class discussions, questionnaire and interviews. The questionnaire was administered to all the 42 student teachers as well as the written tasks were attempted by all the students. The data gathering tools were designed in such a way that they gave an insight into their knowledge of composition of functions, definition of the chain rule, differentiation and application of the chain rule.

Basing on their performance in the written tasks, ten students were sampled out to participate in the interviews. Participants were required to express verbally their responses they gave in the written exercises on their understanding of the chain rule.

4.2 Data Presentation
The written tasks comprised of 8 tasks but only 4 were considered for analysis. These gave a deep insight into the understanding of the chain rule concept. Task problems were testing students on their understanding of the composition of functions, chain rule, computing of derivatives and relationship between compositions of functions and chain rule. In the questionnaire was also a question on the Leibniz notation to see how students’ mental constructions of the chain rule compare to historical understanding of the chain rule.

The questionnaire consisted of 4 items with items 3 and 4 having 6 and 5 sub questions each respectively. The questions on the questionnaire addressed the following concepts and skills: items 1 and 2 focused on the definition of the chain rule and when it is used. Items 3.1 to 3.6 dealt with differentiation and understanding of the chain rule. Basically students checked on whether they could match a particular problem type to an associated method to compute a
given derivative; item 4(i) to 4(v) focused on application of rules of differentiation and the chain rule.

Table 4.1: Distribution of students and skills mastered

<table>
<thead>
<tr>
<th>Definition of chain rule</th>
<th>Use of Product and Quotient rule</th>
<th>Finding Derivative</th>
<th>Understanding and applying Chain rule</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
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<td>11</td>
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<td>2</td>
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<td>6</td>
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<tr>
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<td>1</td>
<td>0</td>
<td>4</td>
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<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>13</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>30</td>
</tr>
</tbody>
</table>

Source: Author compilation

This table is showing distribution of students according to the skills they have mastered. After analysing the results from the written work, questionnaire and interview the following skills were considered in an attempt to offer answers to the research questions: definition, differentiating by product and quotient, evaluating derivatives and differentiating any function by chain rule. From the table, for example, 13 students are able to define the chain rule, 8 are able to use the product and quotient rule, 6 can evaluate derivative of any functions and 3 out of 30 students understand and can apply the chain rule. It shows most of the students are conversant with definition of the chain rule and evaluation of derivatives. Only 3 can apply the chain rule with ease, leaving 27 confessing that they have problems with it.

4.3 Analysis and Discussion

On task 1 which was: Given $f(x) = 3e^{5x}$ and $g(x) = x^5 \cos x$. Find $(fog)$. 
Upon marking the results showed that student teachers experience problems in dealing with composition of functions as out of a total of 30, only 9 showed a complete understanding of the concept. Quite a number of students were not comfortable with decompositions. A good number (16 out of 30) confessed that they did not understand and could not link the composed function with the chain rule and hence failed to come up with correct computations.

Although some students showed evidence that they were able to work with composition of functions, at the object level of APOS, they had problems when it comes to decomposing functions and so thus were relegated to action stage. Found lacking in them was the ability to reverse their thought processes of the previously interiorised actions. The process of linking up composition of functions with derivatives was not complete and as result they could not form the new process of decomposition. This is in line with what happened at the University of Puerto Rico where Rodriguez & Fernandez (2010) noted that students were struggling with identifying the composed functions before applying the chain rule.

On the task \( y = \sin^2 4x \), some students applied the power rule and gave the answer as \( \frac{dy}{dx} = 3\cos^2 4x \). This answer was wrong because the student treated \( y = \sin^2 4x \) as a linear function on which the power rule could be used as in linear algebraic polynomials. This shows that this student is at the action level of mental level of APOS in as far as building knowledge of differentiating functions is concerned. It is important that students understand and recognise situations in which the chain rule could be used to find derivatives.

On scrutinising the students’ written work the researcher noted a number of misconceptions which in turn pointed to how the students constructed their mental images towards understanding of the chain rule. It was noted that even though some students had the notion of the chain rule, still they were not able to apply it. A solution from a selected student can help to show this.
Solution of Student 1

The misconceptions shown by this student is the same as those found by Maharaji (2013) at the University of KwaZulu Natal where in a similar question students avoided the chain rule, opted for the power rule and in the process omitted the square sign of differentiating with respect to the sine function. They were operating at the action stage, just like the student whose solution is provided above.

The task, Find $\frac{dy}{dx}$ if $y = (2 - 3x)^2$, had different presentations from students.

Upon being asked how they differentiated this function in an interview, this is what one student had to say:

**Researcher:** ….now how did you find the derivative of $y$ with respect to $x$ for $y = (2 - 3x)^2$,

**Student:** I brought down the power 2, then multiplied the original function……

**Researcher**... Okay, when do you use the chain rule? Are you aware of the situations when you can use the chain rule?

**Student:** The chain rule is used when……I have two functions multiplying….I mean…. One function written after another…..

**Researcher**…one more question….does the chain rule only work when we have function taken to a power.
Student: ...uum... I’m not so sure... but I think so....

From this interview it is clear that although the student is aware of the power rule they did not fully understand the chain rule. He probably cannot see the connection between the power and chain rule. He is failing to link up composition of function with the chain rule.

Generally it is clear students are not able to transfer the knowledge of the composition of functions and apply the chain rule. Upon analysing the students’ responses from the interview from the written work, it was noted that some students had collections of rules that enabled them to compute derivatives for power functions, trigonometric functions, implicitly defined functions and composite functions. However they showed no evidence that they understand that all these problems are related through the chain rule.

Another student’s presentation on one of the task: \( \text{given } y = \ln(x^2 \sin x), \text{ find } \frac{dy}{dx} \). This one presented his answer as displayed in the extract below.

Solution of Student 2

This is showing that this student was not comfortable with the Leibniz notation. Although the student can conceptualise that \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \), he is failing to perform differentiation on composite functions where he has to find derivatives separately before using the chain rule.

The student upon being asked why he used this method, confessed that, the substitution method and the consequent use of the differentiation rules, was too long a method for him.
He was not comfortable with performing the operations on differentiation from memory. This was evidence enough to confirm that other students like him in class were operating at the action stage of APOS. Yet two other students’ solutions for the same problem was as follows:

Solution of student 3

This result however shows the student recalls that when you are differentiating logarithmic functions such as \( \ln x \) it is equal to \( \frac{1}{x} \) and so upon seeing \( \ln x^2 \sin x \) he quickly thought of expressing the derivative as \( \frac{1}{x^2 - \cos x} \). Thus he has shown serious misconceptions where he has simply differentiated \( \sin 2x \) wrongly again. Thus demonstrating this student was operating at action level. Jojo (2014) in a study carried out at the University of Technology at Kwa Zulu Natal, and had similar results where though the student used correct Leibniz notation of the chain rule he also wrongly differentiated \( y = \ln x \) which was given as \( \frac{dy}{dx} = \frac{1}{\ln x} \). Jojo (2014) concludes that this student is demonstrating a process he is trying to retrieve from memory which is showing the lower level action and process of APOS unlike the student whose solution is shown above.
Solution of student 4

The answers displayed in these presentations are incorrect. Fine the students are quite aware of the Leibniz technique but are uncomfortable with the u-substitution method. Their mental construction are beyond the action level as they do not only know how to perform operations on differentiation from memory or from instruction but they are also demonstrating a process level where they are transforming mental objects perceived as internal and showing some control of the situation. However one of the student was on writing the derivative of

\[ y = \ln u , \quad \Rightarrow \frac{dy}{du} = \frac{1}{u} \]

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (2x^2 \cos x + 2x \sin x + 2x) \]

\[ = \frac{2x^2 \cos x + 2x \sin x + 2x}{u} \]

This one showed how his mental construction of the chain rule are comparing with historical understanding of other concept in as far as using the Leibniz technique is concerned. So his mental construction compares favourably well with historical understanding of the concept.

This student and others also demonstrated conceptual understanding as he could identify and apply differentiation principles. He has interiorised actions since he is able to use the process in an internalised procedure. The written tasks and the questionnaire results showed that a number of students operated at that action stage of APOS where they were only concerned with getting the solution correctly without justification of how they arrived at their answers. It was also clear that some students were just recalling and imitating what they had been
shown and had seen the lecturer do when differentiating. Thus they showed they were operating at a very low level of mental construction of APOS.

The tasks like $y = (2x^2 - 2x)^2$, $y = (2 - 3x)^2$, $y = \ln x^2 \sin x$ gave students a tough time. The written tasks were organised in such a way that they would assist students to have their experiences of reconstructing similar actions as general processes. Problems were presented in such a way that higher order activities of finding derivatives of composition of functions should depend on initial constructed objects like functions and compositions of functions. The previously constructed objects would be organised to tackle problem situations on chain rule. How students constructed their mental structures was revealed in the tasks and class activities given. The exercises also revealed their understanding of functions as well as the chain rule.

In the interviews, questionnaire and written tasks, there were some misconceptions and difficulties emerged from them. These were analysed with the view of exploring and establishing how students construct their different mental structures as they recognise and apply the chain rule.

On analysing the solutions, the researcher was not worried about the correctness of the answer, but rather how they arrived at them. Thus the procedure would help to see the level of the mental construction with regards to understanding the chain rule. Differentiating using the chain rule was done by reconstructing familiar actions though some could apply straight form techniques on problems like $y = (2 - 3x^2)^2$. 
Solution of Student 5.

As shown in the solution some students were guilty of omitting the negative sign outside the bracket after finding the derivatives of the inside function. Some were only multiplying the function by just -3. This clearly showed that students only were aware of the power rule of differentiation but that the chain rule could be used was far-fetched to them. However they were aware of the process of differentiation. The study also showed some students performing some actions which were externally driven by merely recognizing a particular function and then produces a correct answer by possessing just an action conception of the chain rule. This was the case when they attempted the multiple choice questions in the questionnaire.

However overall, on differentiating functions and on this task in particular, the results show that a reasonable number successfully computed this derivative, about 25 out of 42. A vast majority of them were only guilty of making minor errors of omitting the negative sign. This meant they were operating in the object stage of APOS, since they were able to perform the action of differentiating to the process. They also correctly encapsulated the process of differentiation to form objects.
The written tasks also showed that students struggled with linking up of previously learnt algebraic skills like use of brackets where appropriate as well as manipulating algebraic terms in a function. Consequently students would have incorrect presentations of answers, some with negative signs omitted and some trigonometric functions like $y = \sin^2 4x$ treated as linear functions. This was showing a clear sign of an incomplete construction of mental images of using the chain rule. So in a nutshell the written exercises showed these common mistakes when differentiating functions: dropping a negative sign as a simplification error, using right strategies but giving incorrect differentiation.

Though there was evidence that some actions had and were interiorised into processes, the processes in most cases were not encapsulated to objects. Maybe if previous knowledge of manipulating algebraic skills which were just actions had been interiorised, then the situation could be different and complete mental structures could have been shown.

4.4 Conclusion
This chapter looked at data presentation, analysis and discussion of findings. It looked at the results from the questionnaire, written tasks and interview on the students’ understanding of the chain rule. The results have indicated that students have problems in understanding the chain rule concept. It has also been revealed that students build their understanding of the chain rule at different levels as defined by APOS. Current knowledge rests on what was done before and how they form their mental constructions. The next chapter will look at the summary, conclusion and recommendations for further research studies.
CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Introduction
The preceding chapter dealt with data that was qualitatively presented. It also analysed the findings of the study on the student teachers’ understanding of the chain rule. This chapter focuses on summary, conclusion and recommendations on the entire study. Findings of the analysis will be highlighted.

5.2 Summary of findings
This study aimed at exploring student teachers’ understanding of the chain rule concept at Masvingo Teachers’ college. The following findings were revealed by the study. It was revealed that most students can define the chain rule and even evaluate special composite functions. They could recall the chain rule by memorising it. It was also clear that although most of the recall and know the form of the chain rule, they could only calculate derivatives without the conscious use of the chain rule. It was also noted that even though some students had the notion of the chain rule, still they were not able to apply it.

Another finding was that computing derivatives of composite functions requiring application of chain rule was a challenge to many. This exposed their lack of understanding of the concept as they could not link and adjust their steps of operations to differentiate these functions with multiple compositions.

Few students could explain the link or connection between the general statement of the chain rule and what they would have crammed. This was despite the fact that they could provide its general statement and even writing it down. Students could not easily find the relationship between the chain rule and differentiation rules. It was used in isolation and treated as a separate entity.

The study also revealed that students experience difficulties when differentiating a function that require the chain rule.

It was also clear that composite functions are also a problem to handle by students. This concept therefore should be done first before the concept of the chain rule is covered.
The study also revealed that students had misconceptions in notation. They were faced with the dilemma of using Leibniz notation, whether it is a fraction or just single indivisible symbol. In that respect they also had a further dilemma of whether \( \frac{dy}{dx} \) can be cancelled in the equation \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \). This showed that most of the students were performing at action level of understanding. Some students perceived the Leibniz notation as a fraction and simplified the formula. There is need therefore to reassess how students are taught. In cases where there were composite functions, some students deliberately avoided using the Leibniz notation. For instance on evaluating derivatives using the chain rule, the study has revealed that most student teachers recalled the power rule which they memorised for finding derivatives of composite functions. In such questions like \( y = (2 - 3x^2)^3 \), they avoided using the Leibniz notation preferring to use the power rule where in most cases they would make some minor mistakes like omitting the negative sign in the final answers. Probably they could not see the connection between the power and chain rule. They were failing to link up composition of function with the chain rule, thus they were using instructional understanding.

Also revealed in the study was a misconception where students after memorising the power rule applied it even differentiating trigonometric functions which were composite in nature like \( y = \sin^3 4x \). They treated it like a linear function on which the power rule could be used and gave the answer as \( \frac{dy}{dx} = 3\cos^2 4x \), as is done for linear algebraic polynomials. Thus they needed to understand and recognise situations in which the chain rule could be used to find derivatives.

Students’ mental constructions of the chain rule compare favourably well to historical understanding as knowledge of derivatives from Leibniz technique enables them to succeed in computing derivatives. Notation also plays an important part in the success of the use and understanding of the chain rule, just like the role played by notions of composite and derivative functions.

Although to a lesser extent, this study has shown that understanding of composition of functions is also important and key to the learning and understanding of the chain rule. This revelation also concurs with that of Cottrill (1999) which investigated the correlation between students’ understanding of composition of functions and the chain rule.
5.3 Conclusion

The objective of the study was to find out how student teachers learn and understand the chain rule by exploring the mental structures and finding out which APOS levels do they display to understand the concept. Related literature was reviewed and data was collected through questionnaires, written exercises and interviews. So APOS has shown that while students are doing their problem solving on differentiation, they operate at different mental levels but mainly it was discovered that these student teachers are operating at the action and process level. Lecturers therefore need to be aware of this so that they prepare thorough enough on pedagogical strategies which would assist learners to develop appropriate structures. The study showed that APOS theory uses the ACE teaching approach which is a hypothesis of teaching and learning through a cycle which consist of Activities (A), Class discussions (C), and Exercises (E). This is an approach which promotes collaborative learning amongst students and the lecturer or teacher is just there to facilitate the learning process. Instead of employing the traditional lecture style classroom method, it is suggested that students should be pushed past the action and process levels by requiring them to think and explain reasons behind their procedures. The ACE teaching cycle can in this direction and may enable students to reflect on their work and interiorise their actions into process and process into object. Under ACE, activities take place when students work on in class tasks, worksheets and homework. Class discussion gives students the opportunity to reflect on their work and connect ideas and concepts, while exercises outside class give reinforcement to what has been learnt. The approach is effective in assisting students make their mental constructions and learns mathematics as supported by several other researches. Thus the researcher advocates for a paradigm shift from the traditional approach where the lecturer talks and talks and dominate lectures to the constructivist approach where learners are accorded an opportunity to construct their knowledge. The chain rule concept would be better understood if it is taught in this way. The study has shown that if appropriate mental structures are in place students will learn mathematics through having conceptual knowledge and not just learn by rote but by thoughtful and reflective learning. Mathematics should be taught for retention and not for memorisation of facts. Students need to be made aware of the fact that all differentiation rules can be used in conjunction when attempting chain rule problems. They should perceive the rules as entities on which actions can be made. Dubinsky (2010) observed that, in such cases the difficulty depends in the loss of connections between algebraic expressions and the instructions in the situations and not on the nature of the formal expressions.
5.4 Recommendations
Taking into considerations all the findings made in this research, the researcher recommends the following:

• In the teaching and learning of the Chain rule concept, Lectures should place more emphasis on using the Leibniz notation meaningfully so that students’ mental constructions are complete;
• Lecturers should sequence their content in a way that promotes the understanding of the chain rule;
• Instructional strategies should be designed in such a way that encompasses the ACE teaching style;
• APOS should be implemented when teaching the chain rule and other concepts;
• Students should be taught so that they go beyond the process level so that they do not repeat misconceptions held by those in the historical learning of the chain rule.

5.4.1 Recommendations for further areas of study
• Further use of APOS to explore understanding of the connection between composition of functions and the chain rule as well as other concepts;
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KwaZulu-Natal, South Africa.


